CHAPTER 18
GEOMETRIC CONSTRUCTIONS AND SOLID FIGURES

Many ratings in the Navy involve work which requires the construction or subdivision of geometric figures. For example, materials must be cut into desired shapes, perpendicular lines must be drawn, etc. In addition to these skills, some Navy ratings require the ability to recognize various solid figures and calculate their volumes and surface areas.

CONSTRUCTIONS

From the standpoint of geometry, a CONSTRUCTION may involve either the process of building up a figure or that of breaking down a figure into smaller parts. Some typical constructions are listed as follows:

1. Dividing a line into equal segments.
2. Erecting the perpendicular bisector of a line.
3. Erecting a perpendicular at any point on a line.
4. Bisecting an angle.
5. Constructing an angle.
6. Finding the center of a circle.
7. Constructing an ellipse.

EQUAL DIVISIONS ON A LINE

A line may be divided into any desired number of equal segments by the method shown in figure 18-1.

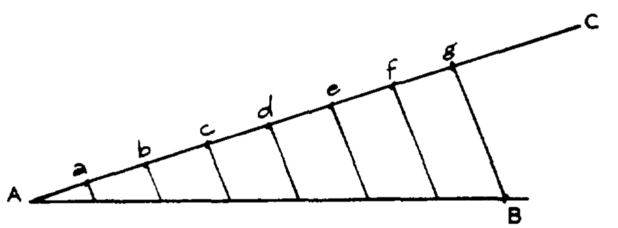


Figure 18-1.—Dividing a line into equal segments.

Suppose that line AB (fig. 18-1) is to be divided into seven equal segments. Draw line AC at any convenient angle with AB and mark off seven spaces of some convenient length, say 1/2 inch, on it. Extend AC, if necessary, in order to get seven intervals of the chosen length on it. This produces the points a, b, c, d, e, f, and g, as shown in figure 18-1. Draw a line from g to B, and then draw lines parallel to gB, starting at each of the points a, b, c, d, e, and f. The segments of AB cut off by these lines are equal in length.

It is frequently necessary to rule a predetermined number of lines on a blank sheet of material. This may be done by a method based on the foregoing discussion. For example, suppose that the sheet of typing paper in figure 18-2 is to be divided into 24 equal spaces.

The 12-inch ruler is laid across the paper at an angle, in such a way that the ends of the

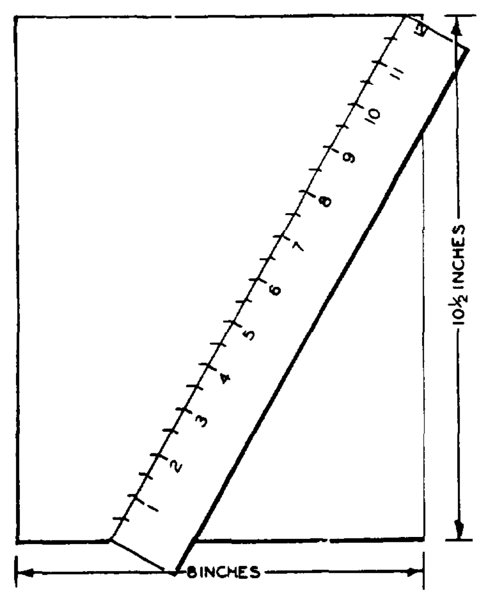


Figure 18-2.—Ruling equal spaces on a sheet of paper.

ruler coincide with the top and bottom edges of the paper. There are 24 spaces, each 1/2 inch wide, on a 12-inch ruler. Therefore, we mark the paper beside each 1/2-inch division marker on the ruler. After removing the ruler, we draw a line through each of the marks on the paper, parallel to the top and bottom edges of the paper.

PERPENDICULAR BISECTOR OF A LINE

To bisect a line or an angle means to divide it into two equal parts. A line may be bisected satisfactorily by measurement, or by a geometric method. If the measuring instrument does not reach the full length of the line, proceed as follows:

1. Starting at one end, measure about half the length of the line and make a mark.
2. Starting at the other end, measure exactly the same distance as before and make a second mark.
3. The bisector of the line lies halfway between these two marks.

The geometric method of bisecting a line is not dependent on measurement. It is based upon the fact that all points equally distant from the ends of a straight line lie on the perpendicular bisector of the line.

Bisecting a line geometrically requires the use of a mathematical compass, which is an instrument for drawing circles and comparing distances. If a line AB is to be bisected as in figure 18-3, the compass is opened until the distance between its points is more than half as long as AB. Then a short arc is drawn above the approximate center of the line and another below, using A as the center of the arcs' circle. (See fig. 18-3.)

Two more short arcs are drawn, one above and one below the approximate center of line AB, this time using B as the center of the arcs' circle.

The two arcs above line AB are extended until they intersect, forming point C, and the two arcs below line AB intersect to form point D. The line joining point C and point D is the perpendicular bisector of line AB.

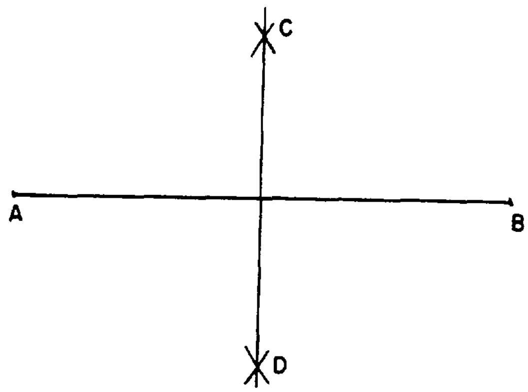


Figure 18-3.—Bisecting a line geometrically.

PERPENDICULAR AT ANY POINT ON A LINE

Figure 18-4 shows a line AB with point C between A and B. A perpendicular to AB is erected at C as follows:

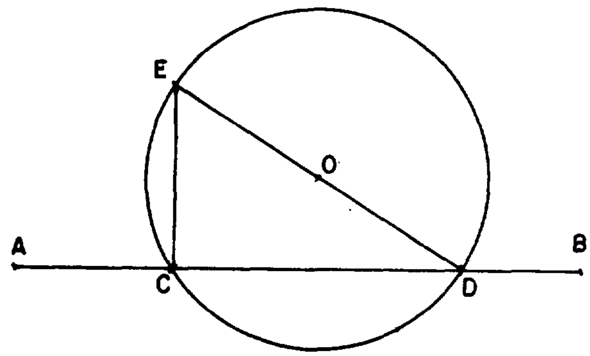


Figure 18-4.—Erecting a perpendicular at a point.

1. Using any convenient point above the line (such as O) as a center, draw a circle with radius OC. This circle cuts AB at C and at D.
2. Draw line DO and extend it to intersect the circle at E.
3. Draw line EC. This line is perpendicular to AB at C.

BISECTING AN ANGLE

Let angle AOB in figure 18-5 be an angle which is to be bisected. Using O as a center and any convenient radius, draw an arc intersecting OA and a second arc intersecting OB. Label these intersections C and D.

Using C and D as centers, and any convenient radius, draw two arcs intersecting halfway between lines OA and OB. A line from O through the intersection of these two arcs is the bisector of angle AOB.

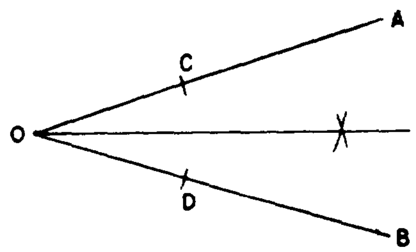


Figure 18-5.—Bisecting an angle.

SPECIAL ANGLES

Several special angles may be constructed by geometric methods, so that an instrument for measuring angles is not necessary in these special cases.

Figure 18-4 illustrates a method of constructing a right angle, DCE, by inscribing a right triangle in a semicircle. But an alternate method is needed for those situations in which drawing circles is inconvenient. The method described herein makes use of a right triangle having its sides in the ratio of 3 to 4 to 5. It is often used in laying out the foundations of buildings. The procedure is as follows:

1. A string is stretched as shown in figure 18-6, forming line AC. The length of AC is 3 feet.
2. A second string is stretched, crossing line AC at A, directly above the point intended as the corner of the foundation. Point D on this line is 4 feet from A.
3. Attach a third string, 5 feet long, at C and D. When AC and AD are spread so that line CD is taut, angle DAC is a right angle.

A 60° angle is constructed as shown in figure 18-7. With AB as a radius and A and B as centers, draw arcs intersecting at C. When A and B are connected to C by straight lines, all three angles of triangle ABC are 60° angles.

The special angles already discussed are used in constructing 45° and 30° angles. A 90° angle is bisected to form two 45° angles, and a 60° angle is bisected to form two 30° angles.

FINDING THE CENTER OF A CIRCLE

It is sometimes necessary to find the center of a circle of which only an arc or a segment is given. (See fig. 18-8.)

From any point on the arc, such as A, draw two chords intersecting the arc in any two points, such as B and C. With the points A, B, and C as centers, use any convenient radius and draw short intersecting arcs to form the perpendicular bisectors of chords AC and AB. Join the intersecting arcs on each side of AC, obtaining line MP, and join the arcs on each side of AB, obtaining line NQ. The intersection of MP and NQ is point O, the center of the circle.

ELLIPSES

An ellipse of specified length and width is constructed as follows:

1. Draw the major axis, AB, and the minor axis, CD, as shown in figure 18-9.
2. On a straightedge or ruler, mark a point (labeled a in the figure) and from this point measure one-half the length of the minor axis and make a second mark (b in figure 18-9). From point a, measure one-half the length of

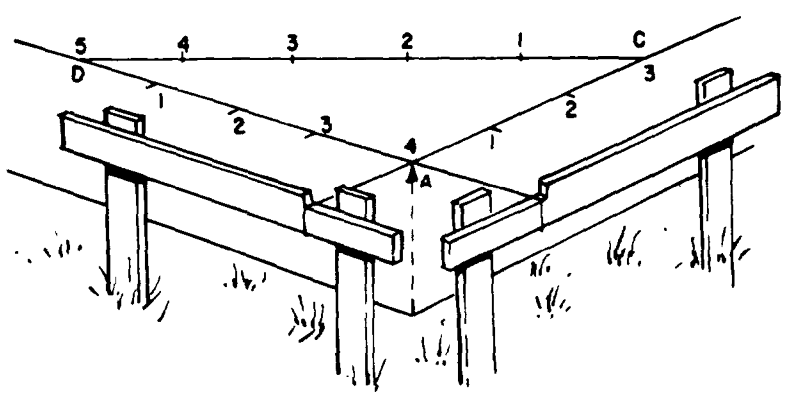


Figure 18-6.—Constructing a right angle by the 3-4-5 method.

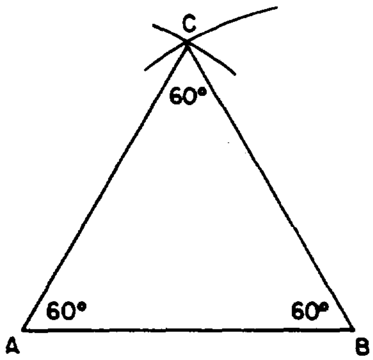


Figure 18-7.—Constructing 60° angles.

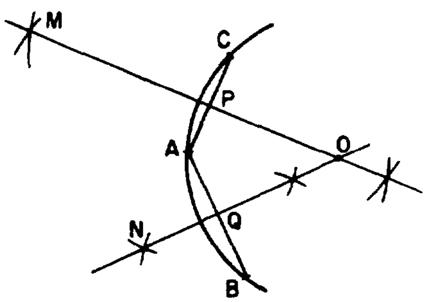


Figure 18-8.—Finding the center of a circle.

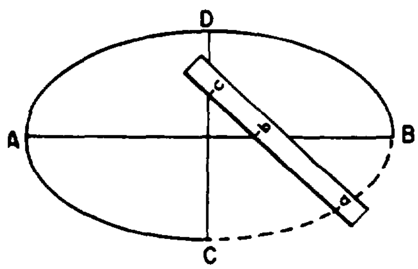


Figure 18-9.—Constructing an ellipse.

the major axis and make a third mark (c in the figure).

3. Place the straightedge on the axes so that b lies on the major axis and c lies on the minor axis. Mark the paper with a dot beside point a. Reposition the straightedge, keeping b on the major axis and c on the minor axis, and make a dot beside the new position of a.

4. After locating enough dots to see the elliptical pattern, join the dots with a smooth curve.

SOLID FIGURES

The plane figures discussed in chapter 17 of this course are combined to form solid figures. For example, three rectangles and two triangles may be combined as shown in figure 18-10. The flat surfaces of the solid figure are its FACES; the top and bottom faces are the BASES, and the faces forming the sides are the LATERAL FACES.

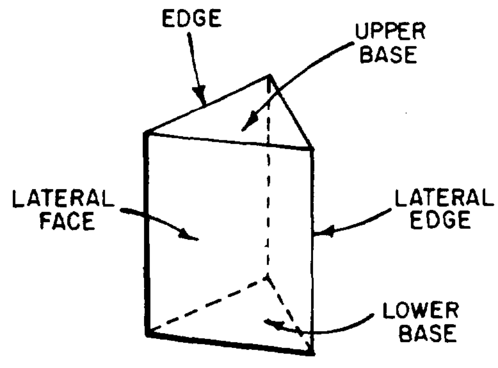


Figure 18-10.—Parts of a solid figure.

Some solid figures do not have any flat faces, and some have a combination of curved surfaces and flat surfaces. Examples of solids with curved surfaces include cylinders, cones, and spheres.

PRISMS

The solid shown in figure 18-10 is a PRISM. A prism is a solid with three or more lateral faces which intersect in parallel lines.

Types of Prisms

The name of a prism depends upon its base polygons. If the bases are triangles, as in figure 18-10, the figure is a TRIANGULAR prism. A RECTANGULAR prism has bases which are rectangles.

If the bases of a prism are perpendicular to the planes forming its lateral faces, the prism is a RIGHT prism.

A PARALLELEPIPED is a prism with parallelograms for bases. Since the bases are parallel to each other, this means that they cut the lateral faces to form parallelograms. Therefore, in a parallelepiped, all of the faces are parallelograms. If a parallelepiped is a right prism, and if its bases are rectangles, it is a rectangular solid. A CUBE is a rectangular solid in which all of the six rectangular faces are squares.

Parts of a Prism

The parts of a prism are shown in figure 18-10. The line formed by the joining of two faces of a prism is an EDGE. If the two faces forming an edge are lateral faces, the edge thus formed is a LATERAL EDGE.

Surface Area and Volume

The SURFACE AREA of a prism is the sum of the areas of all of its faces, including the bases. The VOLUME of a prism may be considered as the sum of the volumes of many thin wafers, each having a thickness of one unit and a shape that duplicates the shape of the base. (See fig. 18-11.)

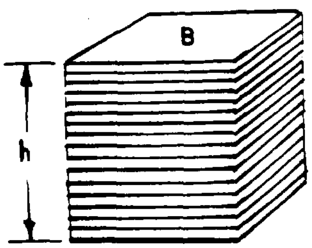


Figure 18-11.—Volume of a prism.

The wafers which comprise the prism in figure 18-11 all have the same area, which is the area of the base. Therefore, the volume of the prism is found by multiplying the area of the base times the number of wafers. Since each wafer is 1 inch thick, the number of wafers is the same as the height of the prism in inches. The resulting formula for the volume of a prism, using B to represent the area of the base and h to represent the height, is as follows:

$$V = Bh$$

When a prism has lateral edges which are not perpendicular to the bases, the height of the prism is the perpendicular distance between the bases. (See fig. 18-12.) The formula for the volume remains the same, even though the prism is no longer a right prism.

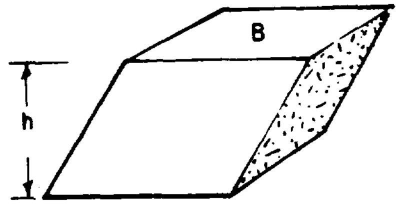


Figure 18-12.—Height of a prism which is not a right prism.

CIRCULAR CYLINDERS

Any surface may be considered as the result of moving a straight line in a direction at right angles to its length. For example, suppose that the stick of charcoal in figure 18-13 is moved from position CD to position AB by dragging it across the paper. The broad mark made by the charcoal represents a plane surface. The surface is said to be "generated" by moving line AB.

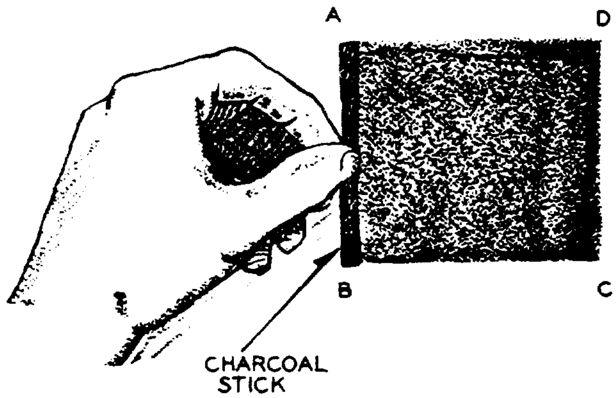


Figure 18-13.—Surface generated by a moving line.

The movement of the line in figure 18-13 may be controlled by requiring that its lower end trace a particular path. For example, if line AB moves so as to trace an ellipse as in figure 18-14 (A), a cylindrical surface is generated by the line. This surface, shown in figure 18-14 (B), is an elliptical cylinder.

Any line in the surface, parallel to the generating line, such as CD or EF in figure 18-14 (B), is an ELEMENT of the cylinder. If the elements are perpendicular to the bases, the cylinder is a RIGHT CYLINDER. If the

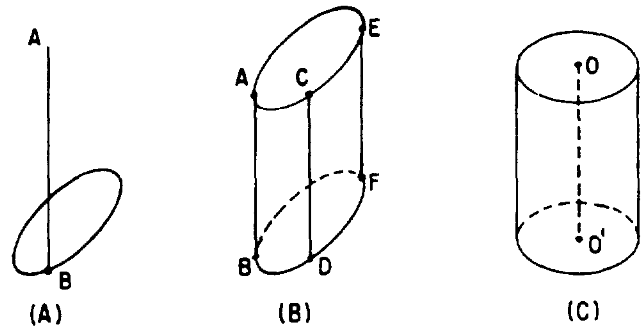


Figure 18-14.—(A) Line generating a cylinder;
(B) elliptical cylinder;
(C) circular cylinder.

ases are circles, the cylinder is a CIRCULAR CYLINDER. Figure 18-14 (C) illustrates a right circular cylinder. Line O-O', joining the centers of the bases of a right circular cylinder, is the AXIS of the cylinder.

Surface Area and Volume

The lateral area of a cylinder is the area of its curved surface, excluding the area of its bases. Figure 18-15 illustrates an experimental method of determining the lateral area of a right circular cylinder.

The card of length L and width W in figure 18-15 is rolled into a cylinder. The height of the cylinder is W and the circumference is L. The lateral area is the same as the original area of the card, LW. Therefore, the lateral area of the cylinder is found by multiplying its height by the circumference of its base. Written as a formula, this is

$$A = Ch$$

EXAMPLE: Find the lateral area of a right circular cylinder whose base has a radius of 4 inches and whose height is 6 inches.

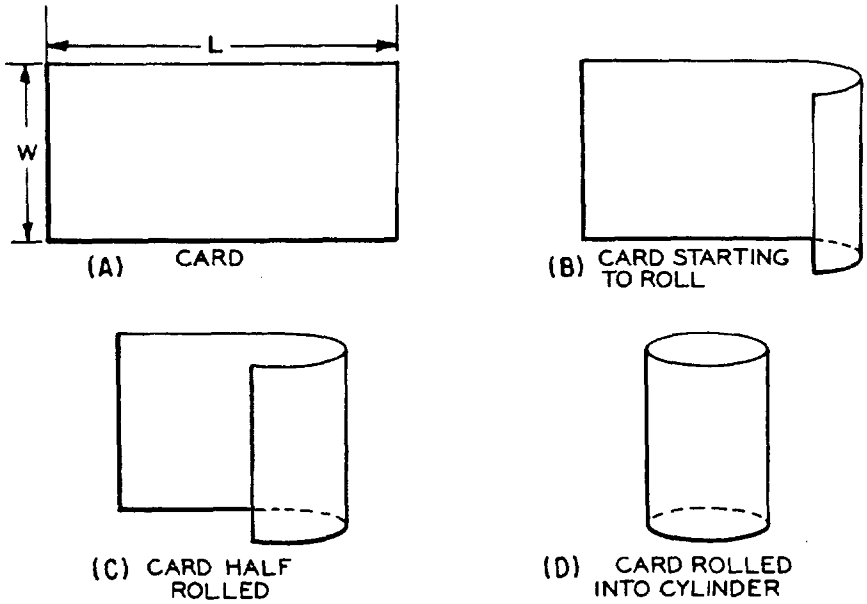


Figure 18-15.—Lateral area of a cylinder.

SOLUTION: The circumference of the base is

$$C = \pi d$$
$$C = 3.14 \times 8 \text{ in.}$$
$$= 25.12 \text{ in.}$$

Therefore,

$$A = 25.12 \text{ in.} \times 6 \text{ in.}$$
$$= 151 \text{ sq in. (approximately)}$$

The formula for the volume of a cylinder is obtained by the same reasoning process that was used for prisms. The cylinder is considered to be composed of many circular wafers, or disks, each one unit thick. The area of each disk, multiplied by the number of disks, is the volume of the cylinder. With V representing volume, A_d representing the area of each disk, and n representing the number of disks, the formula is as follows:

$$V = A_d n$$

Since the number of disks is the same as the height of the cylinder, the formula for the volume of a cylinder is normally written

$$V = Bh$$

In this formula, B is the area of the base and h is the height of the cylinder.

EXAMPLE: Determine the volume of a circular cylinder with a base of radius 5 inches and a height of 14 inches.

SOLUTION:

$$V = Bh$$
$$= (\pi \times 5^2) \times 14$$
$$= 3.14 \times 25 \times 14$$
$$= 78.5 \times 14$$
$$= 1099 \text{ cu in.}$$
$$= 1{,}100 \text{ cu in. (approximately)}$$

Practice problems:

1. Determine the lateral area of a right circular cylinder with a base of diameter 7 inches and a height of 4 inches.

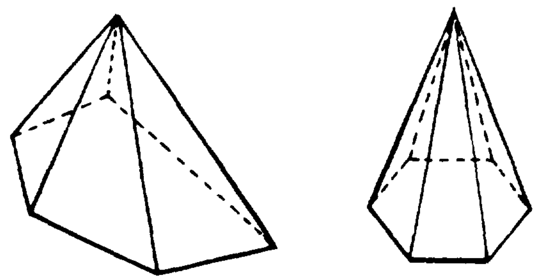
2. Determine the volume of the cylinder in problem 1.

Answers:

1. 88 sq in. 2. 154 cu in.

REGULAR PYRAMIDS AND RIGHT CIRCULAR CONES

A PYRAMID is a solid figure, the lateral faces of which are triangles. (See fig. 18-16.) A REGULAR PYRAMID has all of its lateral faces equal.



(A) (B)

Figure 18-16.—(A) Irregular pyramid; (B) regular pyramid.

A regular pyramid with a very large number of lateral faces would have a base polygon with many sides. If the number of sides is sufficiently large, the base polygon is indistinguishable from a circle and the surface formed by the many lateral faces becomes a smoothly curved surface. The solid figure thus formed is a RIGHT CIRCULAR CONE. (See fig. 18-17.)

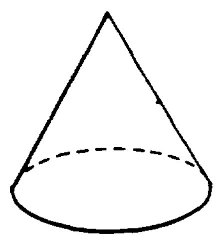


Figure 18-17.—Right circular cone.

Slant Height

The slant height of a regular pyramid is the perpendicular distance from the vertex to the center of any side of the base. For example, the length of line AV in figure 18-18 (A) is the slant height. The slant height of a right circular cone is the length of any straight line joining the vertex to the circumference line of the base. Such a line is perpendicular to a line tangent to the base at the point where the slant height intersects the base. (See fig. 18-18 (B).) Lines AV, BV, and CV in figure 18-18 (B) are all slant heights.

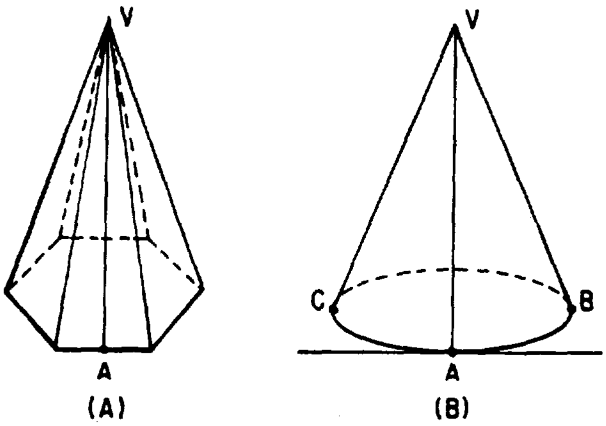


Figure 18-18.—(A) Slant height of a regular pyramid; (B) slant height of a right circular cone.

Lateral Area

The lateral area of a pyramid is the sum of the areas of its lateral faces. If the pyramid is regular, its lateral faces have equal bases; furthermore, the slant height is the altitude of each face. Therefore, the area of each lateral face is one-half the slant height multiplied by the length of one side of the base polygon. Since the sum of these sides is the perimeter of the base, the total lateral area of the pyramid is the product of one-half its slant height multiplied by the perimeter of its base. Using s to represent slant height and P to represent the perimeter of the base, the formula is as follows:

$$\text{Lateral Area} = \frac{1}{2} sP$$

A right circular cone can be considered as a regular pyramid with an infinite number of faces. Therefore, using C to represent the circumference of the base, the formula for the lateral area of a right circular cone is

$$\text{Lateral Area} = \frac{1}{2} sC$$

Volume

The volume of a pyramid is determined by its base and its altitude, as is the case with other solid figures. Experiments show that the volume of any pyramid is one-third of the product of its base and its altitude. This may be stated as a formula with V representing volume, B representing the area of the base, and h representing height (altitude), as follows:

$$V = \frac{1}{3} Bh$$

The formula for the volume of a pyramid does not depend in any way upon the number of faces. Therefore, we use the same formula for the volume of a right circular cone. Since the base is a circle, we replace B with πr^2 (where r is the radius of the base). The formula for the volume of a right circular cone is then

$$V = \frac{1}{3} Bh$$
$$= \frac{1}{3} \pi r^2 h$$

Practice problems:

1. Find the lateral area of a regular pyramid with a 5-sided base measuring 3 inches on each side, if the slant height is 12 inches.

2. Find the lateral area of a right circular cone whose base has a diameter of 6 cm and whose slant height is 14 cm.

3. Find the volume of a regular pyramid with a square base measuring 4 cm on each side, if the vertex is 9 cm above the base.

4. Find the volume of a right circular cone whose base has a diameter of 14 inches, if the altitude is 21 inches.

Answers:

1. 90 sq in.	3. 48 cu cm
2. 132 sq cm	4. 1077 cu in.

SPHERES

A SPHERE is a solid figure with all points on its surface equally distant from its center.

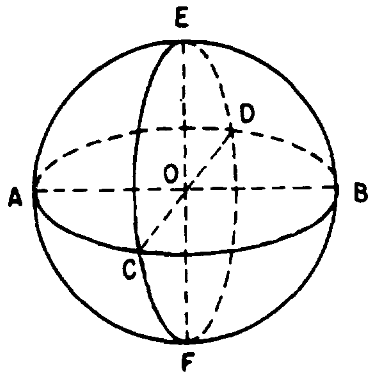


Figure 18-19.—Parts of a sphere.

In figure 18-19, the center of the sphere is point O.

A RADIUS of a sphere is a straight line segment joining the center of the sphere to a point on the surface. Lines OA, OB, OC, OD, OE, and OF in figure 18-19 are radii. A DIAMETER of a sphere is a straight line segment joining two points on the surface and passing through the center of the sphere. Lines AB, CD, and EF in figure 18-19 are diameters. A HEMISPHERE is half of a sphere.

Circles of various sizes may be drawn on the surface of a sphere. The largest circle that may be so drawn is one with a radius equal to the radius of the sphere. Such a circle is a GREAT CIRCLE. In figure 18-19, circles AEBF, ACBD, and CEDF are great circles.

On the surface of a sphere, the shortest distance between two points is an arc of a great circle drawn so that it passes through the two points. This explains the importance of great circles in the science of navigation, since the earth is approximately a sphere.

Surface Area

The surface area of a sphere may be calculated by multiplying 4 times π times the square of the radius. Written as a formula, this is

$$A = 4\pi r^2$$

The formula for the surface area of a sphere may be rewritten as follows:

$$A = (2\pi r)(2r)$$

When the formula is factored in this way, it is easy to see that the surface area of a sphere is simply its circumference times its diameter.

Volume

The volume of a sphere whose radius is r is given by the formula

$$V = \frac{4}{3}\pi r^3$$

EXAMPLE: Find the volume of a sphere whose diameter is 42 inches.

SOLUTION:

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3} \times 3.14 \times (21 \text{ in.})^3 \\ &= \frac{4}{3} \times 3.14 \times 21 \times 21 \times 21 \text{ cu in.} \\ &= 4.187 \times 21 \times 21 \times 21 \text{ cu in.} \\ &= 38{,}776 \text{ cu in. (approximately)} \end{aligned}$$

Practice problems. Calculate the surface area and the volume of the sphere in each of the following problems:

1. Radius = 7 inches
2. Radius = 14 cm

Answers:

1. Area = 615 sq in. (approx.)
   Volume = 1436 cu in. (approx.)
2. Area = 2462 sq cm (approx.)
   Volume = 11,489 cu cm (approx.)